

More on the h -critical numbers of finite abelian groups

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November 21, 2016

Abstract

For a finite abelian group G , a nonempty subset A of G , and a positive integer h , we let hA denote the h -fold sumset of A ; that is, hA is the collection of sums of h not-necessarily-distinct elements of A . Furthermore, for a positive integer s , we set $[0, s]A = \cup_{h=0}^s hA$. We say that A is a generating set of G if there is a positive integer s for which $[0, s]A = G$.

The h -critical number $\chi(G, h)$ of G is defined as the smallest positive integer m for which $hA = G$ holds for every m -subset A of G ; similarly, $\chi(G, [0, s])$ is the smallest positive integer m for which $[0, s]A = G$ holds for every m -subset A of G . We define $\hat{\chi}(G, h)$ as the smallest positive integer m for which $hA = G$ holds for every generating m -subset A of G ; $\hat{\chi}(G, [0, s])$ is defined similarly.

The value of $\chi(G, h)$ has been determined by this author for all G and h , and $\hat{\chi}(G, [0, s])$ was introduced and resolved for some special cases by Klopsch and Lev. Here we determine the remaining two quantities in all cases.

AMS Mathematics Subject Classification:

Primary: 11B75;

Secondary: 05D99, 11B25, 11P70, 20K01.

Key words and phrases:

critical number, abelian groups, sumsets, generating sets.

1 Introduction

Let G be a finite abelian group of order $n \geq 2$, written in additive notation. When G is cyclic, we will identify it with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. More generally, we recall that G has a unique *type* (n_1, \dots, n_r) , where r and n_1, \dots, n_r are positive integers so that $n_1 \geq 2$, n_i is a divisor of n_{i+1} for $i = 1, \dots, r-1$, and

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r};$$

here r is the *rank* of G and n_r is the *exponent* of G .

For a nonempty subset A of G , we let ΣA be the set of all subset sums of A ; that is,

$$\Sigma A = \{\Sigma_{a \in B} a \mid B \subseteq A\}$$

(with the subset sum of the empty-set defined as 0). We then let $\text{cr}(G)$ denote the smallest integer m for which $\Sigma A = G$ holds for all m -subsets A of G ; the smallest integer m for which $\Sigma A = G$ holds for all m -subsets A of $G \setminus \{0\}$ is denoted by $\text{cr}^*(G)$.

The study of *critical numbers* originated with the 1964 paper [7] of Erdős and Heilbronn, in which they asked for $\text{cr}^*(\mathbb{Z}_p)$ for prime values of p . It took nearly half a century, but now, due to the combined results of Diderrich and Mann [6], Diderrich [5], Mann and Wou [14], Dias Da Silva and Hamidoune [4], Gao and Hamidoune [10], Griggs [11], and Freeze, Gao, and Geroldinger [8, 9], we have the critical number of every group:

Theorem 1 (The combined results of authors above) *Suppose that G is an abelian group of order $n \geq 10$, and let p be the smallest prime divisor of n . Then*

$$\text{cr}^*(G) = \text{cr}(G) - 1 = \begin{cases} \lfloor 2\sqrt{n-2} \rfloor & \text{if } G \text{ is cyclic of order } n = p \text{ or } n = pq \text{ where} \\ & q \text{ is prime and } 3 \leq p \leq q \leq p + \lfloor 2\sqrt{p-2} \rfloor + 1^1, \\ n/p + p - 2 & \text{otherwise.} \end{cases}$$

We note that, while it is easy to see that $\text{cr}(G)$ is at least one more than $\text{cr}^*(G)$, there is no obvious reason known for the fact that they differ by exactly one.

In this paper we consider some variations of the critical numbers defined above.

We recall the following definitions. For a positive integer h and a nonempty subset A of G , we let hA denote the *h -fold sumset* of A ; that is, hA is the collection of sums

¹Observe that $\lfloor 2\sqrt{n-2} \rfloor = n/p + p - 1$ in this case.

of h not-necessarily-distinct elements of A . Additionally, for a positive integer s , we set $[0, s]A = \cup_{h=0}^s hA$. Recall also that, for a subset A of G , $\langle A \rangle$ is the subgroup generated by A in G ; that is, $\langle A \rangle$ is the intersection of all subgroups H of G for which $A \subseteq H$. When $\langle A \rangle = G$, we say that A is a *generating set* of G . Clearly, A is a generating set of G if, and only if, there is a positive integer s for which $[0, s]A = G$.

The subject of our paper is the study of the following four quantities:

$$\begin{aligned}\chi(G, h) &= \min\{m : A \subseteq G, |A| \geq m \Rightarrow hA = G\}, \\ \chi(G, [0, s]) &= \min\{m : A \subseteq G, |A| \geq m \Rightarrow [0, s]A = G\}, \\ \widehat{\chi}(G, h) &= \min\{m : A \subseteq G, \langle A \rangle = G, |A| \geq m \Rightarrow hA = G\}, \\ \widehat{\chi}(G, [0, s]) &= \min\{m : A \subseteq G, \langle A \rangle = G, |A| \geq m \Rightarrow [0, s]A = G\}.\end{aligned}$$

It is easy to see that for all G and h we have $hG = G$, so all four quantities are well defined.

The value of $\chi(G, h)$ is now known for every G and h . To state the result, we let $D(n)$ denote the set of positive divisors of the order n of G ; then set

$$f_d(n, h) = \left(\left\lfloor \frac{d-2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{d}$$

for each $d \in D(n)$, and let

$$v(n, h) = \max \{ f_d(n, h) : d \in D(n) \}.$$

We should note that the function $v(n, h)$ has appeared elsewhere in additive combinatorics already. For example, according to the classical result of Diananda and Yap (see [3]), the maximum size of a sum-free set (that is, a set A that is disjoint from $2A$) in the cyclic group \mathbb{Z}_n is given by

$$v(n, 3) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} & \text{if } n \text{ has prime divisors congruent to } 2 \pmod{3}, \\ & \text{and } p \text{ is the smallest such divisor,} \\ \lfloor \frac{n}{3} \rfloor & \text{otherwise;} \end{cases}$$

see e.g. [1] for generalizations.

Our result for $\chi(G, h)$ is then the following:

Theorem 2 (Bajnok; cf. [2]) *For any abelian group G of order n and for all positive integers h we have*

$$\chi(G, h) = v(n, h) + 1.$$

By Theorem 2, the size of the largest *h-incomplete subset* of any group of order n —that is, a subset whose h -fold sumset is not the entire group—equals $v(n, h)$.

Given Theorem 2, the evaluation of $\chi(G, [0, s])$ is immediate. Clearly, $\chi(G, s)$ is an upper bound for $\chi(G, [0, s])$. Suppose then that A is an s -incomplete subset of G of size $\chi(G, s) - 1$. Choose an element $a_0 \in A$, and let

$$B = A - a_0 = \{a - a_0 \mid a \in A\}.$$

Since $|A| = |B|$ and $|sA| = |sB|$, B is also an s -incomplete subset of G of size $\chi(G, s) - 1$. But $[0, s]B = sB$, and thus $[0, s]B \neq G$, which implies that $\chi(G, [0, s])$ is an upper bound for $\chi(G, s)$. Therefore:

Theorem 3 *For any abelian group G of order n and for all positive integers s we have*

$$\chi(G, [0, s]) = \chi(G, s) = v(n, s) + 1.$$

The quantity $\widehat{\chi}(G, [0, s])$ was introduced and investigated by Klopsch and Lev in [12] (though earlier works had treated the case of elementary abelian 2-groups). The value of $\widehat{\chi}(G, [0, s])$ is not known in general. In Section 2 we provide a general lower bound for $\widehat{\chi}(G, [0, s])$, and summarize the main results in a way that allows for comparisons to our other quantities and may generate renewed interest.

Finally, we consider $\widehat{\chi}(G, h)$. In Section 3 we prove that, perhaps surprisingly, $\chi(G, h)$ and $\widehat{\chi}(G, h)$ are always equal:

Theorem 4 *For any abelian group G of order n and for all positive integers h we have*

$$\widehat{\chi}(G, h) = \chi(G, h) = v(n, h) + 1.$$

2 On the value of $\widehat{\chi}(G, [0, s])$

It is an easy exercise to show that, if G is an abelian group of order n that is not isomorphic to \mathbb{Z}_2 , then

$$\widehat{\chi}(G, [0, 1]) = \chi(G, [0, 1]) = v(n, 1) + 1 = n,$$

and if it is not isomorphic to \mathbb{Z}_2 or \mathbb{Z}_2^2 , then

$$\widehat{\chi}(G, [0, 2]) = \chi(G, [0, 2]) = v(n, 2) + 1 = \lfloor n/2 \rfloor + 1.$$

For $s = 3$, the result is considerably more complicated; in particular, it depends on the structure of G and not just on the order n of G :

Theorem 5 (Klopsch and Lev; cf. [12]) *If G is a finite abelian group of order n that is not isomorphic to an elementary abelian 2-group, then*

$$\widehat{\chi}(G, [0, 3]) = \begin{cases} \left(1 + \frac{1}{d}\right) \cdot \frac{n}{3} + 1 & \text{if } G \text{ has a subgroup whose order is congruent to } 2 \pmod{3} \\ & \text{that is not isomorphic to an elementary abelian 2-group,} \\ & \text{and } d \text{ is the minimum size of such a subgroup;} \\ \lfloor \frac{n}{3} \rfloor + 1 & \text{otherwise.} \end{cases}$$

(The value of $\widehat{\chi}(G, [0, s])$ for elementary abelian 2-groups had been determined earlier—see below. We should note that this result appeared in [12] via different expressions.) Theorems 3 and 5 allow for an interesting comparison; for example, we see that

$$\widehat{\chi}(G, [0, 3]) = \chi(G, [0, 3])$$

holds if, and only if, n is odd.

The authors of [12] warn that an expression for $\widehat{\chi}(G, [0, s])$ when $s \geq 4$ “is very difficult, if at all feasible.” One thus may focus on special types of groups. It appears that only two such results are currently known:

Theorem 6 (Klopsch and Lev; cf. [12]) *Let n and s be positive integers. If $n \leq s + 1$, then $\widehat{\chi}(\mathbb{Z}_n, [0, s]) = 1$; otherwise, we have*

$$\widehat{\chi}(\mathbb{Z}_n, [0, s]) = \max \{f_d(n, s) : d \in D(n), d \geq s + 2\} + 1,$$

where $f_d(n, s)$ is as defined above.

Theorem 7 (Lev; cf. [13]) *Let r and s be positive integers, $s \geq 2$. If $r \leq s$, then $\widehat{\chi}(\mathbb{Z}_2^r, [0, s]) = 1$; otherwise we have*

$$\widehat{\chi}(\mathbb{Z}_2^r, [0, s]) = (s + 2) \cdot 2^{r-s-1} + 1.$$

While the two results appear dissimilar, the following general lower bound evaluates to the stated values of $\widehat{\chi}(G, [0, s])$ in both cases.

Proposition 8 *Let G be an abelian group of order n , and let H be a subgroup of G of index $d > 1$ for which G/H is of type (d_1, \dots, d_t) . For each $i = 1, \dots, t$, let c_i be a positive integer with $c_i \leq d_i - 1$, and suppose that*

$$\sum_{i=1}^t \lceil (d_i - 1)/c_i \rceil \geq s + 1.$$

Then we have

$$\widehat{\chi}(G, [0, s]) \geq (1 + \sum_{i=1}^t c_i) \cdot n/d + 1.$$

Before proving Proposition 8, we deduce how it provides exact lower bounds for $\widehat{\chi}(G, [0, s])$ in both Theorems 6 and 7.

Suppose first that G is cyclic and of order n . Clearly, if $n \leq s + 1$, then $\widehat{\chi}(G, [0, s]) = 1$. Assume then that $n \geq s + 2$, and let d be any divisor of n with $d \geq s + 2$. Let H be a subgroup of G of index d , in which case G/H is cyclic and of order d . Then, with $t = 1$ and $c = \lfloor (d - 2)/s \rfloor$, we have $c \geq 1$ and $\lceil (d - 1)/c \rceil \geq s + 1$, so by Proposition 8, we get

$$\widehat{\chi}(G, [0, s]) \geq \left(\left\lfloor \frac{d - 2}{s} \right\rfloor + 1 \right) \cdot \frac{n}{d} + 1 = f_d(n, s) + 1,$$

and our claim follows.

Next, we consider \mathbb{Z}_2^r , the elementary abelian 2-group of rank r . The result is trivial when $r \leq s$, so assume that $s + 1 \leq r$, and let t be an integer with

$$s + 1 \leq t \leq r.$$

Then choosing $H = \mathbb{Z}_2^t$ and $c_i = 1$ for all $i \in \{1, \dots, t\}$, Proposition 8 implies that

$$\widehat{\chi}(\mathbb{Z}_2^r, [0, s]) \geq (t + 1) \cdot 2^{r-t} + 1;$$

in particular, we have

$$\widehat{\chi}(\mathbb{Z}_2^r, [0, s]) \geq (s + 2) \cdot 2^{r-s-1} + 1,$$

as claimed.

Proof of Proposition 8. We shall prove our claim by exhibiting a subset A of G of size

$$|A| = (1 + \sum_{i=1}^t c_i) \cdot n/d$$

for which $\langle A \rangle = G$, but $[0, s]A \neq G$.

Let us identify G/H with

$$K = \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_t},$$

and for $i = 1, \dots, t$, set

$$N_i = \{1, 2, \dots, c_i\} \subseteq \mathbb{Z}_{d_i}.$$

Now let

$$B_i = \{0\}^{i-1} \times N_i \times \{0\}^{t-i}$$

with the understanding that $\{0\}^0$ is to be ignored, and let $B_0 = \{0\}^t$.

Consider $B = \cup_{i=0}^t B_i$. We have $\langle B \rangle = K$, and

$$|B| = 1 + \sum_{i=1}^t c_i.$$

Furthermore, observe that when

$$\sum_{i=1}^t \lceil (d_i - 1)/c_i \rceil \geq s + 1,$$

then $[0, s]B = sB \neq K$.

Now set $A = \pi^{-1}(B)$, where $\pi : G \rightarrow G/H$ is the canonical homomorphism; it is easy to see that A satisfies our requirements. \square

3 The evaluation of $\widehat{\chi}(G, h)$

In this section we prove Theorem 4, namely, that for any abelian group G of order n and for all positive integers h , we have

$$\widehat{\chi}(G, h) = \chi(G, h) = v(n, h) + 1.$$

Note that, obviously,

$$\widehat{\chi}(G, h) \leq \chi(G, h),$$

so by Theorem 2, we have

$$\widehat{\chi}(G, h) \leq v(n, h) + 1.$$

Therefore, to establish Theorem 4, it suffices to find a subset A of G of size $v(n, h)$ for which $\langle A \rangle = G$, but $hA \neq G$.

We proceed by induction on the order n of G . The claim can be easily verified for $n = 2$; we will assume that it also holds for all groups of order at most $n - 1$.

Recall that we have set

$$v(n, h) = \max \{ f_d(n, h) : d \in D(n) \},$$

where $D(n)$ is the set of positive divisors of n , and

$$f_d(n, h) = \left(\left\lfloor \frac{d-2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{d}.$$

We consider two cases.

Case 1: There is a $d_0 \in D(n) \setminus \{n\}$ for which $v(n, h) = f_{d_0}(n, h)$.

Let H be a subgroup of index d_0 in G . Then G/H has order $d_0 < n$, so by our inductive hypotheses, it contains a subset B of size $v(d_0, h)$ for which $\langle B \rangle = G/H$, but $hB \neq G/H$.

Let $\pi : G \rightarrow G/H$ denote the canonical homomorphism, and let $A = \pi^{-1}(B)$. Then $\langle A \rangle = G$, $hA \neq G$, and the size of A is $v(n, h)$, since

$$|A| = |B| \cdot |H| = v(d_0, h) \cdot \frac{n}{d_0} \geq f_{d_0}(d_0, h) \cdot \frac{n}{d_0} = \left(\left\lfloor \frac{d_0 - 2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{d_0} = f_{d_0}(n, h) = v(n, h).$$

This completes the proof of Case 1.

Case 2: For all $d \in D(n) \setminus \{n\}$,

$$f_d(n, h) \leq v(n, h) - 1.$$

We then must have

$$v(n, h) = f_n(n, h) = \left\lfloor \frac{n - 2}{h} \right\rfloor + 1.$$

Let us consider first the case when n equals a prime number p ; we will then identify G with the cyclic group \mathbb{Z}_p . Let

$$A = \left\{ 1, 2, \dots, \left\lfloor \frac{p - 2}{h} \right\rfloor + 1 \right\} \subseteq \mathbb{Z}_p.$$

Clearly, A has size $v(p, h)$, and $\langle A \rangle = \mathbb{Z}_p$. Moreover,

$$hA = \left\{ h, h + 1, \dots, h \cdot \left\lfloor \frac{p - 2}{h} \right\rfloor + h \right\},$$

from which we see that $h - 1 \notin hA$, and thus $hA \neq \mathbb{Z}_p$. Therefore, our claim holds for prime values of n .

Assume now that n is composite. We will show that for each prime divisor p of n , $p - 1$ must be divisible by h .

To do so, let p be a prime divisor of n , and let

$$p - 2 = ch + r$$

for some (unique) integers c and r with

$$0 \leq r \leq h - 1.$$

Note that n is composite, so $p < n$. Therefore, by assumption, we have

$$f_p(n, h) \leq v(n, h) - 1 = \left\lfloor \frac{n - 2}{h} \right\rfloor.$$

Here

$$f_p(n, h) = \left(\left\lfloor \frac{p-2}{h} \right\rfloor + 1 \right) \cdot \frac{n}{p} = \left(\frac{p-2-r}{h} + 1 \right) \cdot \frac{n}{p} = \frac{n}{h} + \frac{h-2-r}{h} \cdot \frac{n}{p},$$

which is more than $\left\lfloor \frac{n-2}{h} \right\rfloor$, unless $r = h-1$. But then

$$p-2 = ch + h-1,$$

so $p-1$ is divisible by h , as claimed.

This implies that every positive divisor of n is congruent to 1 mod h ; in particular, so is n , and thus

$$v(n, h) = f_n(n, h) = \left\lfloor \frac{n-2}{h} \right\rfloor + 1 = \frac{n-1}{h}.$$

We thus need to find a subset A of G of size $\frac{n-1}{h}$ for which $\langle A \rangle = G$, but $hA \neq G$.

Suppose that G has rank r and that it is of type (n_1, \dots, n_r) ; that is,

$$G \cong \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}$$

for integers n_1, \dots, n_r for which $n_1 \geq 2$ and n_i is a divisor of n_{i+1} for each $i = 1, \dots, r-1$. As we just proved, $n_i - 1$ is divisible by h for each $i = 1, \dots, r$.

We construct A as follows: for each $i = 1, \dots, r$, let

$$N_i = \left\{ 1, 2, \dots, \frac{n_i-1}{h} \right\} \subseteq \mathbb{Z}_{n_i}$$

and

$$A_i = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_{i-1}} \times N_i \times \{0\}^{r-i};$$

we then set

$$A = \cup_{i=1}^r A_i.$$

Then

$$|A_i| = n_1 \dots n_{i-1} \cdot \frac{n_i-1}{h};$$

since the r sets are pairwise disjoint, the size of A equals

$$|A| = \sum_{i=1}^r |A_i| = \sum_{i=1}^r n_1 \dots n_{i-1} \cdot \frac{n_i-1}{h} = \frac{n_1 \dots n_r - 1}{h} = \frac{n-1}{h} = v(n, h).$$

Furthermore, since $n_r - 1$ is divisible by h , we have $n_r \geq h+1$, and thus

$$A \supseteq A_r \supseteq \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_{r-1}} \times \{1\},$$

which implies that $\langle A \rangle = G$.

Finally, we show that $hA \neq G$ by verifying that $0 \notin hA$. Let a_1, \dots, a_h be (not-necessarily distinct) elements of A . For each $i = 1, \dots, r$, there is a unique $k_i \in \{1, \dots, r\}$ for which $a_i \in A_{k_i}$; let

$$k = \max\{k_i \mid i = 1, \dots, r\}.$$

But then the k -th coordinate of $a_1 + \dots + a_h$ (as an element of $\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}$) is at least 1 and at most $n_k - 1$, so

$$a_1 + \dots + a_h \neq 0.$$

This completes our proof. \square

References

- [1] B. Bajnok, On the maximum size of a (k, l) -sum-free subset of an abelian group. *Int. J. Number Theory* **5**(6) (2009), 953–971.
- [2] B. Bajnok, The h -critical number of finite abelian groups. *Unif. Distrib. Theory* **10**, no.2 (2015), 93–15.
- [3] P. H. Diananda and H. P. Yap, Maximal sum-free sets of elements of finite groups. *Proceedings of the Japan Academy*, **45** (1969) 1–5.
- [4] J. A. Dias Da Silva and Y. o. Hamidoune, Cyclic space for Grassmann derivatives and additive theory. *Bull. London Math. Soc.*, **26** (1994) 140–146.
- [5] G. T. Diderrich, An Addition Theorem for Abelian Groups of Order pq , *J. Number Theory*, **7** (1975) 33–48.
- [6] G. T. Diderrich and H. B. Mann, Combinatorial Problems in Finite Abelian Groups. *A Survey of Combinatorial Theory*, J. N. Srivastava et al., ed., North-Holland (1973).
- [7] P. Erdős and H. Heilbronn, On the addition of residue classes (mod p). *Acta Arith.*, **9** (1964) 149–159.
- [8] M. Freeze, W. Gao, and A. Geroldinger, The critical number of finite abelian groups. *J. Number Theory*, **129** (2009) 2766–2777.
- [9] M. Freeze, W. Gao, and A. Geroldinger, Corrigendum to “The critical number of finite abelian groups. *J. Number Theory*, **129** (2009) 2766–2777”, *J. Number Theory*, **152** (2015) 205–207.

- [10] W. Gao and Y. O. Hamidoune, On additive bases. *Acta Arithmetica*, **88**:3 (1999) 233–237.
- [11] J. R. Griggs, Spanning subset sums for Finite Abelian groups, *Discrete Mathematics*, **229** (2001) 89–99.
- [12] B. Klopsch and V. F. Lev, Generating abelian groups by addition only. *Forum Math.*, **21**:1, (2009) 23–41.
- [13] V. F. Lev, Generating binary spaces. *J. Combin. Theory, Ser. A* **102** (2003) 94–109.
- [14] H. B. Mann and Y. F. Wou, Addition theorem for the elementary abelian group of type (p, p) , *Monatshefte für Math.* **102** (1986) 273–308.